Math 275D Lecture 26 Notes

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1 Stochastic Differential Equations

1.1 Examples of SDEs

If we have a process satisfying a stochastic differential equation, we may want to recover a concrete description of it. We want to solve **stochastic differential equations** of the form

$$dX_t = f(t, X_t)\, dt + g(t, X_t)\, dB_t.$$

Example 1.1. Consider the equation

$$dX_t = \mu X_t\, dt + \sigma X_t\, dB_t,$$

where μ, δ are fixed.

To solve this, we hope that $X_t = f(t, B_t)$. If this is correct, we get that

$$f_t = \frac{1}{2} f_{xx} = \mu f, \qquad f_x = \sigma f.$$

We can then solve this equation to get a solution of the form

$$f = e^{\sigma x + g(t)}.$$

Plugging this back into the equation, we get

$$f = e^{\sigma x + (\mu - \sigma/2)t}.$$

So the solution to the SDE is

$$X_t = e^{\sigma B_t + (\mu - \sigma/2)t} \cdot X_0.$$

These do not always have this kind of solution. Here is an example:

Example 1.2. Consider the **Ornstein-Uhlenbeck process** satisfying the equation

$$dX_t = -aX_t\,dt + \sigma\,dB_t.$$

If we try to solve it the same way, we get

$$f_t + \frac{1}{2}f_{xx} = -af, \qquad f_x = \sigma.$$

Then we get

$$f = \sigma x + g(t), \qquad g' = -a(\sigma x + g(t)).$$

So there are no solutions of this type to this differential equation.

To solve this, recall the trick to solving the ODE $f' = -af + g(x)$. We introduce a factor like $e^{\int a}$. The idea is to introduce a factor

$$Y = g(t) \cdot X(t).$$

Then we get

$$\begin{aligned} dY &= g'X(t)\,dt + g(t) \cdot dX_t \\ &= g'X_t\,dt + g(t)(-aX_t)\,dt + g(t)\sigma\,dt. \end{aligned}$$

We want something of the form $g' = ag$, so $g = ce^{at}$. With this g, we get

$$dY = \sigma\,dB_t \implies Y = Y_0 + \sigma B_t$$

We then get

$$X = ce^{-at}(Y_0 + \sigma B_t).$$

Remark 1.1. This process has $\mathbb{E}[X_t^2] < \infty$, so it has a limit.

Example 1.3. Let's say we have a **Brownian bridge**, a Brownian motion with $B_1 = 0$. This event has probability 0, so we can't condition on $B_1 = 0$. We can set up the equation

$$X_t = -\frac{X_t}{1-t}\,dt + \,dB_t.$$

We can solve this as before with $a(t) = -\frac{1}{1-t}$. We have $g' = a(t)g$, so $e^{\int_0^t a(s)\,ds} \cdot C$. And for $Y = g \cdot X$, we get

$$Y = g\,dB_t = \frac{1}{1-t}\,dB_t.$$

So

$$X = (1-t)\int_0^t \frac{1}{1-s}\,dB_s.$$

gives us the formula for a Brownian bridge.

1.2 Existence and uniqueness of solutions to SDEs

Theorem 1.1. *Consider the equation*

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dB_t.$$

Suppose μ, σ *are smooth, Lipschitz:*

$$|\mu(t,x) - \mu(t,y)| \leq K|x-y|, \qquad |\sigma(t,x) - \sigma(t,y)| \leq L|x-y|,$$

and $\mu\sigma \leq (1+|x|)C$*. Then there is a unique solution to this equation.*

Proof. To prove uniqueness, suppose

$$X_t = \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s + X_0,$$

$$Y_t = \int_0^t \mu(s, Y_s)\, ds + \int_0^t \sigma(s, Y_s)\, dB_s + Y_0.$$

By the Lipschitz condition, we get that $X_t - Y_t$ is bounded by the integral of something like $X_s - Y_s$. That is, the L^∞ norm is bounded by the L^1 norm. Such a function must be 0. In fact, we actually bound $\mathbb{E}[(X_t - Y_t)^2]$ like this.

For existence, take a sequence of iterates

$$X_t^{(n+1)} = \int_0^t \mu(s, X_s^{(n)})\, ds + \int_0^t \sigma(s, X_s^{(n)})\, dB_s + C_0,$$

and show that this converges. We do this by using the contraction mapping theorem: we show that

$$\|X^{(n+1)} - X^{(n)} \leq C\|X^{(n)} - X^{(n-1)}\|$$

for some $C < 1$. To get $C < 1$, we make the time interval small enough. □